Exercise 46

Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \le x < 3 \\ 2x - a + b & \text{if } x \ge 3 \end{cases}$$

Solution

$\frac{x^2-4}{x-2}$ is a rational function that is continuous on $x < 2$, $ax^2 - bx + 3$ is a polynomial that is continuous on $2 \le x < 3$, and $2x - a + b$ is a polynomial that is continuous on $x \ge 3$. Determine a and b by requiring the function to be continuous at $x = 2$ and $x = 3$.

$$\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^+} f(x) \\ \lim_{x\to 2^-} \frac{x^2-4}{x-2} &= \lim_{x\to 2^+} (ax^2 - bx + 3) \\ \lim_{x\to 2^-} \frac{(x+2)(x-2)}{x-2} &= a(2)^2 - b(2) + 3 \\ \lim_{x\to 2^-} (x+2) &= 4a - 2b + 3 \\ 4 &= 4a - 2b + 3 \\ 1 &= 4a - 2b \end{aligned}$$

$$\begin{aligned} \lim_{x\to 3^-} f(x) &= \lim_{x\to 3^+} f(x) \\ \lim_{x\to 3^-} (ax^2 - bx + 3) &= \lim_{x\to 3^+} (2x - a + b) \\ a(3)^2 - b(3) + 3 &= 2(3) - a + b \\ 9a - 3b + 3 &= 6 - a + b \\ 10a - 4b + 3 &= 6 \\ 10a - 4b &= 3 \end{aligned}$$

This is a system of two equations for two unknowns, a and b.

$$\left.\begin{aligned} 10a - 4b &= 3 \\ 4a - 2b &= 1 \end{aligned}\right\}$$

$$\left.\begin{aligned} 10a - 4b &= 3 \\ 8a - 4b &= 2 \end{aligned}\right\}$$

Subtract the respective sides to eliminate b.

$$2a = 1$$

Therefore,

$$a = \frac{1}{2}.$$

Plug this result back into either of the two equations.

$$10a - 4b = 3$$

$$10\left(\frac{1}{2}\right) - 4b = 3$$

$$5 - 4b = 3$$

$$4b = 2$$

Therefore,

$$b = \frac{1}{2}.$$

Below is a graph of $f(x)$ versus x with $a = \frac{1}{2}$ and $b = \frac{1}{2}$.